Hopf bifurcation and chaos analysis of a discrete – delay dynamic model for a stock market

Loretti I. Dobrescu∗, Dumitru Opris†
∗ *University of Padua, Department of Economics, 33 via del Santo, Padua, 35123 Italy* , † *West University of Timisoara, Department of Applied Mathematics, Bd. V. Parvan, nr. 4, Timisoara, 300223 Romania*
E-mail: lorettiisabella.dobrescu@unipd.it, opris@math.uvt.ro

Version: February 27, 2008

The time evolution of prices and saving in a stock market is modelled by a discrete-delay nonlinear dynamical system. The proposed model has a unique and unstable steady-state, so that the time evolution is determined by the nonlinear effects acting out the equilibrium. The analysis of linear approximation through the study of the eigenvalues of the Jacobian matrix is carried out in order to characterize the local stability property and the local bifurcations in the parameter space. If the delay is equal to zero, Lyapunov exponents are calculated. For certain values of the model parameters we prove that the system has a chaotic behaviour. Some numerical examples are finally given for justifying the theoretical results.

Key Words: dynamic models, bifurcation, Lyapunov exponents, stock market.

1. INTRODUCTION

One of the cornerstones of the modern theory of finance is the view that asset prices exhibit erratic behavior. In this paper, we consider a discrete - delay time deterministic model, which describe the interactions between the price index of a stock market and the net stock of savings collected by the mutual fund. The model is based on the assumption that there are two different kinds of economic agents interacting in the market: the "dealers", who are directly admitted to the securities negotiation, and the "savers" who intend to invest in the stock market but, being scarcely informed, prefer to underwrite shares of a mutual fund. The paper proceeds as following: Section 2 describes the economic motivation and the general structure of the model, taking into account the savings from the moment $n-m$. We will show that for each economically feasible set of parameters, the model has a unique steady state. In Section 3 we will analyze the characteristic equation and the value of the bifurcation for parameter a, when we don't have a delay and when we do. Section 4 presents the normal form and in Section 5 we will offer numerical example for fixed parameters. In Section 6, for $m = 0$, we present an algorithm for assigning the Lyapunov exponents. For certain values of the parameters, it results that the first Lyapunov exponent is positive. Consequently, the system behaves chaotically. Using a program in Maple 11, we visualize the orbits of the state variables. Section 7 concludes.

2. THE DISCRETE-DELAY MODEL FOR A STOCK MARKET

We assume the day as the unit of measurement for time. This allows us to give the following simple description of the rules which regulate the evolution of the price index, p, in time and the net stock of savings collected by the fund, s. If the price level is p_n and the savings collected by the funds is s_{n-m} at times $n-m$, $m \geq 0$, then at time $n+1$, the stock market, where only the dealer participates to the negotiations, will open with a new value of the index p_{n+1}, determined by a law of the kind:

$$p_{n+1} - p_n = g\left(s_{n-m}, p_n\right) \tag{1}$$

Afterwards, the stock market being closed, the savers, who act by underwriting shares of the mutual fund or asking for the repayment of the already held ones, will buy or sell, and such choices give rise to the new value of the variable s, through a law of the kind:

$$s_{n+1} - s_n = f\left(s_{n-m}, p_{n+1}, p_{n+1} - p_n\right) \tag{2}$$

The two functions, g and f, are supposed to be at least $\mathbb{C}^1$ and to satisfy the following assumptions [1.]: 1.) $\frac{\partial g}{\partial s_{n-m}} > 0$; 2.) $\frac{\partial g}{\partial p_n} < 0$; 3.) $\frac{\partial f}{\partial s_{n-m}} < 0$; 4.) $\frac{\partial f}{\partial p_{n+1}} < 0$; 5.) $\frac{\partial f}{\partial (p_n - p_{n+1})} < 0$. It is easy to verify [1.], since the previous five hypotheses are sufficient to ensure the uniqueness of the equilibrium, if it exists.

Such equilibrium values, say $\overline{s}$ and $\overline{p}$, can be considered "natural levels", which may be accepted thought as solutions of a system of equations or deduced from some general macroeconomic considerations, that the agents perceive as reference value to which they compare the present situation in order to take the investment decision. Under these assumptions the model can be rewritten as:

$$\begin{aligned} s_{n+1} - s_n &= F\left(s_{n-m} - \overline{s}, p_{n+1} - \overline{p}, p_{n+1} - p_n\right) \\ p_{n+1} - p_n &= G\left(s_{n-m} - \overline{s}, p_n - \overline{p}\right) \end{aligned} \tag{3}$$

whose dynamics depend on the differences from the values that agents perceive as natural. The functions F and G satisfy the same first-order conditions defined for f and g, but in this context it is natural to claim that $G(0,0) = F(0,0,0) = 0$. After the following change of variables:

$$S_n = s_n - \overline{s}, \qquad S_{n-m} = s_{n-m} - \overline{s}, \qquad P_n = p_n - \overline{p} \tag{4}$$

we obtain the model:

$$\begin{aligned} S_{n+1} - S_n &= F\left(S_{n-m}, P_{n+1}, P_{n+1} - P_n\right) \\ P_{n+1} - P_n &= G\left(S_{n-m}, P_n\right) \end{aligned} \tag{5}$$

Moreover, to approach [1.], as a consequence of some further assumptions about the prevailing behavior of the agents, we specify the map in a polynomial form given by:

$$\begin{aligned} F(x,y,z) &= Ez - Ay - Bx^3 \\ G(x,y) &= Cx - Dy^3 \end{aligned}$$

The system is given by:

$$\begin{array}{rcl} S_{n+1}-S_n & = & -AP_{n+1}-BS_{n-m}^3+E(P_{n+1}-P_n) \\ P_{n+1}-P_n & = & CS_{n-m}-DP_n^3 \end{array} \quad (6)$$

where all the coefficients A, B, C, D, E, whose meaning can be easily deduced from the previous discussion about the assumptions 1.)...5.), are real and positive.

Noting $S_{n-m} = x^1, ...S_n = x^{m+1}, P_n = x^{m+2}$, the time evolution of the system (6) is obtained by the iteration of the $m+2$-dimensional map, defined by:

$$\begin{pmatrix} x^1 \\ ... \\ x^m \\ x^{m+1} \\ x^{m+2} \end{pmatrix} \rightarrow \begin{pmatrix} x^2 \\ ... \\ x^{m+1} \\ cex^1-b(x^1)^3+x^{m+1}-ax^{m+2}-de(x^{m+2})^3 \\ cx^1+x^{m+2}-d(x^{m+2})^3 \end{pmatrix} \quad (7)$$

and

$$a = A, b = B, c = C, d = D, e = E - A \quad (8)$$

According to the above discussion, the parameters a, b, c and d are positive, whereas the coefficient e can take non-negative values provided that $e + a > 0$. For $m = 0$, the system (6) is analyzed in Bischi and Valori [1]. We study, the conditions satisfied by the parameters a, b, c, d so that the system (6) may accept a closed and stable curve in the neighborhood of the fixed point $(0, 0, .., 0) \in \mathbb{R}^{m+2}$.

3. THE ANALYSIS OF THE CHARACTERISTIC EQUATION FOR (7)

As usual, the first step in the qualitative analysis of a dynamic model given by (6) is the localization of the steady states of the $m+2$-dimensional map, given by (7). In our case, using the method from [3.] and [5.], the following result holds:

PROPOSITION 1. *(i). For each economically feasible set of parameters, the $m+2$ -dimensional map, given by (7), has the unique steady $O = (0, 0, .., 0) \in \mathbb{R}^{m+2}$.*

(ii). If $m > 1$,the Jacobi matrix of (7) computed at O is:

$$A = \begin{pmatrix} 0 & 1 & ... & 0 & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & 1 & 0 \\ ce & 0 & ... & 1 & -a \\ c & 0 & ... & 0 & 1 \end{pmatrix}, \quad (9)$$

and if $m = 0$, the Jacobi matrix of (7) computed at O is:

$$A = \begin{pmatrix} 1+ec & -a \\ c & 1 \end{pmatrix}. \quad (10)$$

(ii). If $m = 0$, the characteristic equation is given by:

$$\lambda^2 - (2 + ce)\lambda + 1 + ce + ac = 0, \quad (11)$$

and if $m \geq 0$,the characteristic equation is given by:

$$\lambda^m(\lambda - 1)^2 - ce(\lambda - 1) + ac = 0. \quad (12)$$

In the following analysis we shall consider fixed, positive values of the parameters b, c, d and we shall investigate the effect of changes of the parameters $e \in \mathbb{R}$ and $a \in \mathbb{R}$. The choice of the parameters a and e as bifurcation parameters is related to the fact that they give a measure of the two opposite forces which determine the relative weight of the agent's attitude to realize the capital gains, measured by the parameter a, and their speculative attitude, measured by the parameter e.

If $m = 0$, using the method from Kuznetsov [5], the following results hold:

Proposition 2. *(i). If $c > 0$, $e \in \left(-\frac{4}{c}, 0\right) - \left\{-\frac{3}{c}, -\frac{2}{c}\right\}$ and $a = a_0 = -e$, then the characteristic equation has two eigenvalues, and there exists a unit circle, given by $\lambda_1(a_0) = exp(i\theta(a_0)), \lambda_2(a_0) = \lambda_1(a_0)$, where $\theta(a_0) = \arccos\left(\frac{2+ce}{2}\right)$.*

(ii). Let the variable transformation:

$$a(\beta) = a_0 + \frac{(1+\beta)^2 - 1}{c} \tag{13}$$

where $|\beta|$ is sufficiently small.

With (13), the characteristic equation (11) becomes:

$$\mu^2 - (2+ce)\mu + (1+\beta)^2 = 0. \tag{14}$$

The eigenvalues of equation (14) are given by:

$$\mu_1(\beta) = (1+\beta)\exp(i\omega(\beta)), \mu_2(\beta) = \overline{\mu_1(\beta)}, \tag{15}$$

where

$$\omega(\beta) = \arccos\left(\frac{2+ce}{2(1+\beta)}\right). \tag{16}$$

(iii). If $\mu = \mu_1(\beta)$ is the eigenvalue of (14), the eigenvectors $q \in \mathbb{R}^2$, $p \in \mathbb{R}^2$ corresponding to A, and A^T respectively, have the following components::

$$q_1 = 1, q_2 = \frac{1+ec-\mu}{a}, p_1 = \frac{1-\overline{\mu}}{2(1-\overline{\mu})+ec}, p_2 = \frac{a}{2(1-\overline{\mu})+ec}p_1 \tag{17}$$

where $a = a(\beta)$.

From Proposition 2, it results that $\mu_1(0) = \mu_1(a_0)$, and all the assumption for the occurrence of a Neimark-Sacker hold, for the parameter β and, thus, also for a_0.

If $m = 1$, using the method from Ford and Wulf [3], the following result holds:

Proposition 3. *(i). If $c > 0$, $e \in \left(-\frac{1}{c}, 0\right)$ and $a = a_0$, where*

$$a_0 = \frac{1 - ec - \sqrt{1+ec}}{c}, \tag{18}$$

then the characteristic equation has two eigenvalues, and there exists a unit circle, given by $\lambda_1(a_0), \lambda_2(a_0) = \lambda_1(a_0)$ and an eigenvalue $\lambda_3(a_0)$ with absolute value less than one, where:

$$\lambda_1(a_0) = \exp\left(\frac{2 - c(a_0+e)}{2}\right), \lambda_3(a_0) = -c(a_0+e). \tag{19}$$

PROPOSITION 4. *(ii) For $|\beta|$ sufficiently small, let the transformation of the variable:*

$$a(\beta) = a_0 + f(\beta), \tag{20}$$

where:

$$f(\beta) = \frac{(1+\beta)^2(1-\sqrt{(1+\beta)^2+ec}) - (1-\sqrt{1+ec})}{c}. \tag{21}$$

With (20), the characteristic equation (12) becomes:

$$\mu^3 - 2\mu^2 + (1-ec)\mu + c(a_0 + f(\beta) + e) = 0 \tag{22}$$

The eigenvalues of equation (22) are given by:

$$\begin{aligned} \mu_1(\beta) &= (1+\beta)\exp(i\omega(\beta)), \mu_2(\beta) = \overline{\mu_1(\beta)}, \\ \mu_3(\beta) &= -\frac{c(a_0 + f(\beta) + e)}{(1+\beta)^2} \end{aligned} \tag{23}$$

where:

$$\omega(\beta) = \arccos\left(\frac{2(1+\beta)^2 - c(a_0 + f(\beta) + e)}{2(1+\beta)^2}\right). \tag{24}$$

(iii) If $\mu = \mu_1(\beta)$ is the eigenvalue of (22), the eigenvectors $q \in \mathbb{R}^3$, $p \in \mathbb{R}^3$ corresponding to A, and A^T respectively, have the following components::

$$\begin{aligned} q_1 &= 1, q_2 = \mu, q_3 = \frac{ec + \mu(1-\mu)}{a}, \\ p_1 &= \frac{(\overline{\mu}-1)(\mu-1)^2}{\mu(\overline{\mu}-1)(\mu-1)^2 - (ec+1-\overline{\mu})(\overline{\mu}-1)}, p_2 = \frac{1}{\overline{\mu}}p_1, \end{aligned} \tag{25}$$

$$p_3 = \frac{1}{(\overline{\mu}-1)\overline{\mu}^2}p_1. \tag{26}$$

From Proposition 3, it results that $\mu_1(0) = \lambda_1(a_0)$, $\mu_3(0) = \lambda_3(a_0)$ and all the assumption for the occurrence of a Neimark-Sacker hold, for the parameter β and, thus, also for a_0.

4. THE NORMAL FORM OF THE MAP (7)

In order to write the normal form of the application (7), we take into account that the right member of (7), contains only terms of the first and third order and we apply the method from [6.].

If $m = 0$, the following result holds:

PROPOSITION 5. *(i). The normal form for (7) is given by:*

$$z_{n+1} = \mu(\beta)z_n + \frac{1}{6}g_{30}z_n^3 + \frac{1}{2}g_{21}z_n^2\overline{z_n} + \frac{1}{2}g_{12}z_n\overline{z_n}^2 + \frac{1}{6}g_{03}\overline{z_n}^3 \tag{27}$$

where $\mu(\beta)$ is given by equation (15), $z_n \in \mathbb{C}^2$ and:

$$\begin{aligned} g_{30} &= g_{30}(\beta) = -6bp_1 - 6d(p_1e + p_2)q_2^3, \\ g_{21} &= g_{21}(\beta) = -6bp_1 - 6d(p_1e + p_2)q_2\overline{q_2}^2, \\ g_{12} &= g_{12}(\beta) = -6bp_1 - 6d(p_1e + p_2)q_2^2\overline{q_2}, \\ g_{03} &= g_{03}(\beta) = -6bp_1 - 6d(p_1e + p_2)\overline{q_2}^3 \end{aligned} \tag{28}$$

where p_1, p_2, q_2 are given by (17).

(ii). Let us consider $l_1(0) = \text{Re}(\exp(-i\omega(0)g_{21}(0)))$, where $\omega(0)$ is given by (16). The condition for a supercritical bifurcation is $l_1(0) < 0$.

(iii). The orbits of system (6), is given by:

$$S_n = z_n + \overline{z_n}, P_n = q_2 z_n + \overline{q_2 z_n} \tag{29}$$

where z_n is a solution of equation (26).

If $m = 1$, using the method from Mircea *et al.* [6], the following result holds:

PROPOSITION 6. *(i). The normal form associated to (7) yields:*

$$z_{n+1} = \mu(\beta) z_n + \frac{1}{2} g_{21}(\beta) z_n^2 \overline{z_n} \tag{30}$$

where $\mu(\beta)$ is given by equation (23), $z_n \in \mathbb{C}^2$ and:

$$g_{21} = g_{21}(\beta) = -6bp_1 - 6d(p_1 e + p_3) q_3 \overline{q_3}^2$$

where p_1, p_2, q_3 are given by (25).

(ii). Let us consider $l_1(0) = \text{Re}(\exp(-i\omega(0)g_{21}(0)))$, where $\omega(0)$ is given by (24). The condition for a supercritical bifurcation is $l_1(0) < 0$.

(iii). The orbits of system (6), is given by:

$$\begin{aligned} S_n &= q_2 z_n + \overline{q_2 z_n} + k_1 \mu_3(\beta)^n, \\ P_n &= q_3 z_n + \overline{q_3 z_n} + k_1 \left(\frac{ec + \mu_3(\beta) - \mu_3(\beta)^2}{a} \right)^n \\ S_{n-m} &= U_n = z_n + \overline{z_n} + k_1 \end{aligned} \tag{31}$$

where z_n is a solution of equation (29), $\mu_3(\beta)$ is given by equation (23) and $k_1 \in \mathbb{R}$.

5. NUMERICAL EXAMPLE

Using a program in Maple 11, for $m = 0$ and $b = 0.5, c = 0.4$, $d = 0.1$, $e = -2$, $\beta = -0.001$, $l_1(0) = 0.2449$, $n = 1300$, just after the Neimark-Hopf bifurcation, a trajectory is numerically generated starting from an initial condition close to the fixed point O, the value of the bifurcation is $a_0 = -e$; the trajectory is represented in the phase plane (Sn, Pn) in Fig.1. For $m = 1$ and $b = 0.5$, $c = 0.4$, $d = 0.1$, $e = -2$, $\beta = -0.001$, $l1(0) = 0.2449$, $n = 1300$, just after the Neimark-Hopf bifurcation, a trajectory is numerically generated, starting from an initial condition close to the fixed point O, the value of the bifurcation is $a_0 = 3.3819$; the trajectory is represented in the phase plane (Sn, Pn) in Fig.2 and in the phase plane (Un, Sn) in Fig.3.

The value of the bifurcation is different if $m = 0$ or if $m = 1$. For the case $m = 2$, a similar analysis can be conducted.

6. THE LYAPUNOV EXPONENT FOR THE SYSTEM (16) WITH M= 0

In this section we analyze the behaviour of system (6) solutions for $m = 0$ and $A < 0, B < 0, C > 0, D < 0, E < 0$. For these values, we calculate the Lyapunov exponents. If $x_n = S_n$ and $y_n = P_n$, the system (6) for $m = 0$ is given by:

$$\begin{aligned} x_{n+1} &= (1 - C(A - E))x_n - Ay_n - Bx_n^3 + DAy_n^3 \\ y_{n+1} &= Cx_n + y_n - Dy_n^3 \end{aligned} \tag{32}$$

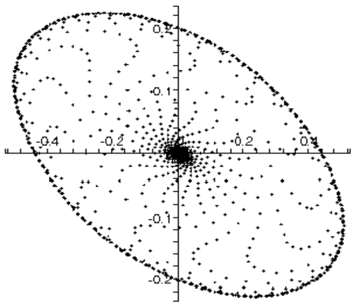

Fig. 1. The trajectory in the phase plane (Sn, Pn)

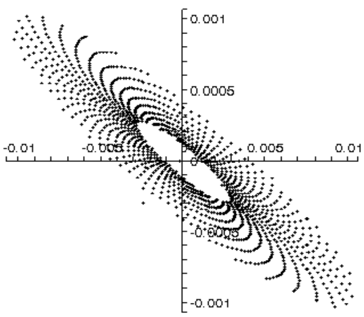

Fig 2. The trajectory in the phase plane (S_n, P_n)

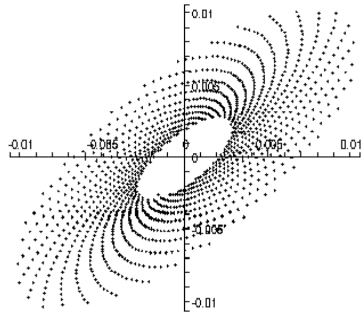

Fig 3. The trajectory in the phase plane (U_n, S_n)

Using the method from Janaki and Rangarajan [4], The Lyapunov exponents can be obtained by solving the system (32) and the system:

$$
\begin{aligned}
z_{n+1} &= \arctan\left(\frac{F_1(x)}{F_2(x)}\right) \qquad (33)\\
\lambda_{n+1} &= \lambda_n + \ln\left(|\cos z_n \cos z_{n+1}(f_{11} - f_{21}\tan z_{n+1}) - \sin z_n \cos z_{n+1}(f_{12} - f_{22}\tan z_{n+1})|\right)\\
\mu_{n+1} &= \mu_n + \ln\left(|\sin z_n \cos z_{n+1}(f_{11}\tan z_{n+1} + f_{21}) + \cos z_n \cos z_{n+1}(f_{12}\tan z_{n+1} + f_{22})|\right)
\end{aligned}
$$

where:

$$
\begin{aligned}
f_{11} &= \frac{\partial f_1}{\partial y_1}, f_{12} = \frac{\partial f_1}{\partial y_2}, f_{21} = \frac{\partial f_2}{\partial y_1}, f_{22} = \frac{\partial f_2}{\partial y_2}\\
f_1 &= (1 - C(A - E))y_1 - Ay_2 - By_1^3 + DAy_2^3\\
f_2 &= Cy_1 + y_2 - Dy_2^3\\
F_1(x) &= f_{22}\sin z_n - f_{21}\cos z_n\\
F_2(x) &= f_{11}\cos z_n - f_{12}\sin z_n
\end{aligned}
$$

The Lyapunov exponents are:

$$\lambda = \lim_{n\to\infty} \frac{\lambda_n}{n}, \mu = \lim_{n\to\infty} \frac{\mu_n}{n} \tag{34}$$

For the following parameter values $A = 0.4$, $B = -0.3$, $C = 0.2$, $D = -0.1$, $E = 0.2$, the Lyapunov exponents are $\lambda \cong 0.047, \mu = -0.178$. Because $\lambda > 0$, it follows that the system (32) is chaotic. A similar analysis can be conducted for $m = 1$.

7. CONCLUSIONS

This paper developed a deterministic model for the description of the time evolution and interactions between savings and price level in a stock market. The model is not based on the market microstructure and on the optimizing behavior of the agents, but it gives a fairly general description of the main (nonlinear) interactions between two kinds of agents that we assume are acting in two different sections of the market: the dealers (administrators of mutual funds, directly admitted to the securities negotiation, and the savers who, after taking their investment decision, buy or sell shares of the mutual funds. The model describes the situation when we take into account savings collected by funds s_{n-m} at the time $n-m$, for $m = 0$ and $m = 1$. Considering parameter s as the bifurcation parameter, the normal form of the model is obtained. Knowing its solution we described the dynamics of the model mentioned . We showed that for certain values of the model parameters, the system display a chaotic behavior due to the fact that the first Lyapunov exponent is positive.

REFERENCES

[1] Bischi, G.B., Valori, V., Nonlinear effects in a discrete-time dynamic model of a stock market, Chaos, Solitons and Fractals 11 (2000), pag. 2103-2121.

[2] Dobrescu, L.I., Opris, D., 'Neimark-Sacker bifurcation for the discrete-delay Kaldor Model', Chaos, Solitons & Fractals (2007), doi:10.1016/j.chaos.2007.10.044.

[3] Ford, N.J., Wulf, V., Numerical Hopf bifurcation for the delay logistic equation, MCCM Technical Report Nr. 322, Manchester University, 1998.

[4] Janaki, T.M., Rangarajan, G., Computation of the Lyapunov spectrum for continuous-time dynamical systems and discrete maps, Physical Review E Volume 60, Number 6.

[5] Kuznetsov, Y.A., Elements of Applied Bifurcation Theory, Applied Mathematical Sciences 112, Springer-Verlag, New York, 1995.

[6] Mircea, G., Neamtu, M., Opris, D., Dynamic systems in economics, mechanics and biology described by differential equations with time delay, Ed. Mirton, Timisoara, 2003.