

Chaotic Phenomena at Czech Capital Market

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Abstract: Several financial applications of methods developed originally in physics are demonstrated. First, the time development of cumulative strain prior to material failures and preceding log-periodic oscillations have been observed in Telecom stock prices. Second, the search for chaotic patterns in PX index time series and its volatility was performed using correlation dimensions and BDS test. Finally, Hurst exponent computed for PX index was slightly greater than 0.5, which supported persistent behaviour with tendency to cycles creation.

Keywords: Log-periodic oscillations, Stock prices, Correlation dimension, Hurst exponent

1. Log-periodic oscillations

First, let us mention very important notion of scale invariance [5]. Under a change of scale, say $x \rightarrow \lambda x$, scale invariance of a system means that a function $f(x)$ reproduces itself, i.e.

$$f(x) = \mu f(\lambda x) \Rightarrow f(x) = C x^\alpha$$

Solving this equation with respect to α , we arrive at

$$\mu \lambda^\alpha = 1 = \exp(2\pi ni) = \cos(2\pi n) \quad \alpha_n = -\frac{\ln \mu}{\ln \lambda} + i \frac{2\pi n}{\ln \lambda}$$

On the other hand, it was found from many studies of material failures (ruptures, occurrence of earthquakes etc.), the cumulative strain $\varepsilon(t)$ depends on the stress acting as [5], [6]

$$\varepsilon(t) = A + B(t_c - t)^\alpha \left[1 + \sum_{n=1}^{\infty} c_n \cos\left(\frac{2\pi n}{\ln \lambda} \ln(t_c - t)\right) \right] \quad A > 0, B < 0, 0 < \alpha < 1$$

where t_c is a critical time, when a failure occurs. It means, cumulative strain is accompanied by log-oscillations, which can serve as failure predictors. Sornette transferred these ideas also to financial crashes [5]. Keeping only the leading term in log-oscillations, we modify the first equation to

$$\ln P(t) = A + B(t_c - t)^\alpha \{1 + C \cos[\omega \ln(t_c - t) - \phi]\}$$

where $P(t)$ denotes stock prices and ϕ is phase shift.

We have applied this tool to the time series of Telecom stock prices during the period of the growth from 21/10/1999 to 24/03/2000 (107 daily

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values). After final date, price decrease has been observed for nearly 400 days. Using regression analysis, we estimated parameters as follows

$$\begin{aligned} A &= 6.88757 & B &= -0.02458 & \alpha &= 0.73746 \\ C &= 0.13346 & \omega &= 4.55392 & \phi &= -0.42953 \end{aligned}$$

The results obtained signalize the improvement of the model using log-oscillations: model MSE decreased from 0.212 (with $C = 0$) to 0.060 (with $C = 0.13346$).

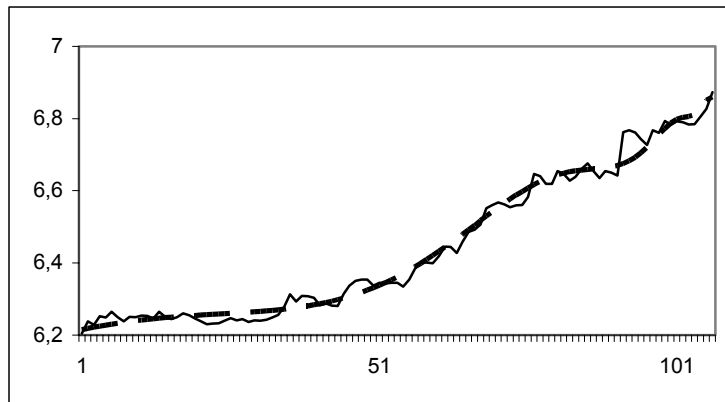


Figure 1. Log-periodic model applied to TELECOM stock prices

2. Chaotic behaviour

The reasons for the construction of quantitative characteristics of chaotic behavior are:

- quantifiers can help to distinguish deterministic chaos from „noisy“ behavior, produced by the action of external random influences
- quantifiers can help to determine minimum number of variables needed for the construction of a dynamical model of the system
- quantifiers can help to classify systems according to universally valid regularities
- changes of quantifiers may signalize changes in qualitative behavior of a system

On principle, we can use two kinds of description. The first type stresses the dynamics of chaotic behavior and corresponding quantifiers describe the system evolution and mutual position of neighbouring trajectories. The second type is based on the geometry of trajectories in state space. Here we shall confine ourselves to the first kind of the description. One of characteristic features of chaotic behavior is the divergence of neighbouring trajectories. Let us consider two close points $x_0, x_0 + \varepsilon$.

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Applying certain iterative function n -times, the distance between these points will be

$$d_n = \left| f^{(n)}(x_0 + \varepsilon) - f^{(n)}(x_0) \right|$$

In the case of chaotic behavior, we expect the exponential growth of this distance with the number of iterations

$$d_n / \varepsilon = \exp(\lambda n) \quad \lambda = \frac{1}{n} \ln(d_n / \varepsilon)$$

In limit approaching $\varepsilon \rightarrow 0$ we obtain after small algebra

$$\lambda = \frac{1}{n} \left\{ \ln |f'(x_0)| + \ln |f'(x_1)| + \dots + \ln |f'(x_n)| \right\}$$

This *Lyapunov exponent* characterizes the speed of the divergence of neighbouring trajectories [1]. It is given as averaged natural logarithm of absolute value of the derivatives of iterative function at individual points of a trajectory. One-dimensional iterative function has chaotic trajectories, if average Lyapunov exponent is positive (the condition of divergence).

Having a time series of equidistant values $x_0, x_1, x_2, \dots, x_n$, we can determine Lyapunov exponent using the following approach. Let us take two trajectories with starting points x_i, x_j and let us create the sequence of differences

$$d_0 = |x_j - x_i| \quad d_1 = |x_{j+1} - x_{i+1}| \quad \dots \quad d_n = |x_{j+n} - x_{i+n}|$$

We shall assume time evolution in the form

$$d_n = d_0 \exp(\lambda n) \quad \Rightarrow \quad \lambda = \frac{1}{n} \ln \frac{d_n}{d_0}$$

For $\lambda > 0$, the behavior of trajectories is chaotic. However, generally, λ depends on the choice of x_i . Therefore, the more reliable is to compute *average Lyapunov coefficient* from large number N initial values regularly distributed over whole attractor.

Grassberger and Procaccia [3] introduced the characteristic called *correlation dimension*, based on the behavior of so-called correlation sum. For the computation of correlation dimension, we need data about the evolution of a trajectory (in sum n values). For each i -th point of the trajectory, we seek relative frequency $p_i(r)$ of trajectory points, lying at distance less than r from the point i (except i -th point)

$$p_i(r) = n_i / (n-1)$$

Correlation sum is then computed as average relative frequency

$$C_1(r) = \frac{1}{n} \sum_{i=1}^n p_i(r)$$

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Obviously $C_1(r)=0$, if r is less than minimal distance among the points of a trajectory. On contrary $C_1(r)=1$ means, the distances among individual points do not exceed r . Minimal possible non-zero value $C_1(r)=2/(n(n-1))$ occurs in the case, only one distance is less than r . Relative frequency can be formally expressed using Heaviside step function

$$p_i(r) = \frac{1}{n-1} \sum_{j=1, j \neq i}^n H(r - |x_i - x_j|)$$

Similarly, correlation sum can be written as

$$C_1(r) = \frac{1}{n(n-1)} \sum_{i=1}^n \sum_{j=1, j \neq i}^n H(r - |x_i - x_j|)$$

In limit case $n \rightarrow \infty$, correlation sum is melted into *correlation integral*. Correlation dimension is then given by formula

$$D_1 = \lim_{r \rightarrow 0} \frac{\log C_1(r)}{\log r}$$

A time series of single variable can be often sufficient for the determination of important characteristics of a multidimensional dynamical system. The groups of values

$$x_{t+1}, x_{t+2}, \dots, x_{t+d} \quad t = 0, 1, 2, \dots, (n-d)$$

give the coordinates of a point in d -dimensional space. Then the sequence of these groups describes the time evolution of a system in d -dimensional *embedding space*. In this case, correlation sum can be written as

$$C_d(r) = \frac{1}{n(n-1)} \sum_{i=1}^n \sum_{j=1, j \neq i}^n H(r - |x_i - x_j|)$$

because it depends on embedded dimension d . Vector \mathbf{x}_i of dimension d has components

$$\mathbf{x}_i = (x_i, x_{i+L}, x_{i+2L}, \dots, x_{i+(d-1)L})$$

where L is time lag between neighbouring values. The length of the difference of two vectors is mostly calculated as Euklid distance

$$(\mathbf{x}_i - \mathbf{x}_j) = \sqrt{\sum_{k=0}^{d-1} (x_{i+kL} - x_{j+kL})^2}$$

Then it holds for correlation dimension

$$D_d = \lim_{r \rightarrow 0} \frac{\log C_d(r)}{\log r}$$

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Grassberger and Procaccia have studied the behavior of these characteristic [3]. In the case of i.i.d. (independent identically distribution] with regular distribution is

$$D_1 = \lim_{r \rightarrow 0} \frac{\log C_1(r)}{\log r} = \lim_{r \rightarrow 0} \frac{\log 2 + \log r}{\log r} = 1$$

$$D_2 = \lim_{r \rightarrow 0} \frac{\log C_2(r)}{\log r} = \lim_{r \rightarrow 0} \frac{\log 4 + 2 \log r}{\log r} = 2$$

and generally $D_d = d$. On the contrary, in the case of non-linear deterministic process is the behavior of correlation sum quite different. For example, in the case of roof mapping is $D_1 = D_2 = \dots = D_d = 1$. This tool was used also for our time series of PX index logarithmic returns and correlation dimensions for different embedded dimensions were computed:

| length | d=1 | d=2 | d=3 | d=4 | d=5 | d=6 |
|-------------|-------|-------|-------|-------|-------|-------|
| 225 | 1.032 | 1.915 | 2.176 | 4.232 | 5.132 | - |
| 450 | 1.036 | 2.099 | 2.822 | 4.148 | - | 5.382 |
| 675 | 1.052 | 2.091 | 2.995 | 3.999 | 4.720 | - |
| 900 | 1.016 | 2.105 | 3.069 | 4.010 | 4.790 | - |
| 1125 | 1.021 | 2.101 | 3.060 | 3.962 | 4.637 | - |
| 1350 | 1.014 | 2.122 | 3.057 | 3.886 | - | 5.013 |
| 1575 | 1.013 | 2.080 | 3.037 | 3.992 | - | 5.078 |
| 1800 | 1.025 | 2.082 | 3.003 | 3.917 | 4.573 | 4.913 |
| 2025 | 1.033 | 2.072 | 3.048 | 3.916 | 4.532 | 4.865 |
| 2250 | 1.034 | 2.078 | 3.022 | 3.892 | 4.549 | 4.918 |

Table 1. PX index logarithmic returns: Correlation dimensions computed for different embedded dimensions (d) and different lengths of time series

Clearly, up to $d=5$, $D_d \approx d$ and the behaviour of PX returns seems to be similar to i.i.d. process. On the other hand, for $d>5$, $D_d < d$. Thus, if this time series is chaotic, it should be generated by some high-dimensional process.

Brock, Dechert and Scheinkman have shown, for a finite r and i.i.d. process the following relation is valid

$$C_d(r) = [C_1(r)]^d$$

and suggested test statistic [2]

$$T_d(r, n) = \frac{C_d(r, n) - [C_1(r, n)]^d}{s_d(r, n)}$$

where $C_d(r, n)$, $C_1(r, n)$ are sample correlation sums and $s_d(r, n)$ is the estimate of the standard deviation of the difference in the previous

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expression . This statistic has asymptotically standard normal distribution $N(0,1)$ providing the validity of null hypothesis (i.i.d. process).

| length | d=2 | d=3 | d=4 | d=5 | d=6 |
|--------|------|-------|-------|-------|-------|
| 225 | - | 3.02 | 3.89 | 4.36 | 5.21 |
| 450 | 3.00 | 3.80 | 4.52 | 5.15 | 6.40 |
| 675 | 4.56 | 5.67 | 6.37 | 6.94 | 8.23 |
| 900 | 3.69 | 4.53 | 5.45 | 6.07 | 7.08 |
| 1125 | 4.65 | 6.35 | 7.52 | 8.38 | 9.53 |
| 1350 | 5.49 | 7.43 | 8.66 | 9.63 | 10.86 |
| 1575 | 6.03 | 8.04 | 9.42 | 10.48 | 11.64 |
| 1800 | 7.08 | 9.46 | 11.14 | 12.48 | 14.03 |
| 2025 | 8.12 | 10.82 | 12.59 | 13.99 | 15.62 |
| 2250 | 8.68 | 11.40 | 13.36 | 14.90 | 16.67 |

Table 2. PX logarithmic returns: BDS test statistic computed for different embedded dimensions (d) and different lengths of time series

All test statistics given in the Table 2 exceed the critical value 1.96 and null hypothesis (i.i.d. process) is rejected in all cases.

3. Rescaled range analysis

The general procedure is the following [4]:

- given a time series containing values x_1, x_2, \dots, x_N registered in time points $t = 1, 2, \dots, N$; thus, N time intervals with unit length arise
- subdivide the whole time series into m neighbouring intervals with length n , $N = mn$
- compute average value for each interval

$$\bar{x}_j = \frac{1}{n} \sum_{i=1}^n x_{ij} \quad j = 1, 2, \dots, m$$

- create the time series of cumulative deviations from average value

$$z_{kj} = \sum_{i=1}^k (x_{ij} - \bar{x}_j) \quad k = 1, 2, \dots, n$$

- compute the range of cumulated deviations for each interval

$$R_j = \max(z_{kj}) - \min(z_{kj}) \geq 0$$

- compute standard deviation for each interval

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$$s_j = \sqrt{\frac{1}{n} \sum_{i=1}^n (x_{ij} - \bar{x}_j)^2}$$

- compute rescaled range R_j / s_j for each interval
- average value R/S for an interval with length n is then

$$(R/S)_n = \frac{1}{m} \sum_{j=1}^m (R_j / S_j)$$

Hurst assumed general type of the dependence of R/S on time in the form [4]

$$(R/S)_n = Cn^H$$

where C is a constant and H *Hurst exponent*. In practice, Hurst exponent can be estimated using linear regression

$$\log [(R/S)_n] = \log C + H \log n$$

by virtue of the values $(R/S)_n$ computed for different n . Interpretation of Hurst exponent is the following: If $H = 0.50$, then is time series generated by i.i.d. process. In the range $0.50 < H < 1.00$ are corresponding processes called *persistent* and they are characterized by long memory. In the range $0 < H < 0.50$ so-called *antipersistent* processes are underlying, which alter signs more frequently than white noise.

As for PX logarithmic returns, Hurst exponent was computed for different lengths; the results are given in the Table 3:

| | | | | | |
|---------------|-------------|-------------|-------------|-------------|-------------|
| length | 225 | 450 | 675 | 900 | 1125 |
| H | 0.598 | 0.592 | 0.565 | 0.561 | 0.546 |
| length | 1350 | 1575 | 1800 | 2025 | 2250 |
| H | 0.552 | 0.558 | 0.556 | 0.552 | 0.548 |

Table 3. PX index logarithmic returns: Hurst exponent computed for different lengths of time series

All values of Hurst exponent are greater than 0.5 and weak tendency to persistent behaviour is proved in all segments of time series investigated.

4. Conclusions

With the use of real financial data, we demonstrated the power of certain tools originally developed in physics. First, log-periodic oscillations were detected prior to Telecom price decrease in March 2000. Corresponding regression model exhibited clear improvement after including this phenomena. Further, we investigated the possibility of chaotic behaviour in

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financial time series. If this is true, then only high-order chaos probably arises, because correlation dimensions computed for PX index are of the same order as embedded dimensions up to 5-th order. On the other hand, BDS test results clearly signalize departure from purely random behaviour. Moreover, the Hurst exponent computed for this time series is slightly greater than 0.5; thus, some kind of persistent behaviour is present and we expect weak tendency to cycles creation.

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