Chaotic Waterwheel: Discrete vs Continuous Mass Representation

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Abstract. The equations of chaotic waterwheel dynamics are derived through analytical mechanics using the Lagrange approach. Discrete and continuous mass cases are considered, the latter leading to Lorenzian system. The dynamics is showed to be different in these two cases, with discrete mass dynamics matching better the Lorenzian one as the number of buckets increases.
Keywords: Lagrange Approach, Continuous Mass Approximation, Lorenzian Dynamics.

1 Introduction

A mechanical rotating waterwheel model was proposed and built by Willem Malkus to demonstrate unpredictable nature of physical systems [4]. The Malkus' waterwheel dynamics closely resembles the dynamics of famous Lorenz system [3], and, therefore, can be viewed as its mechanical analogy.

The physical waterwheel is simple in its conception, yet not completely intuitive in its performance. A constant flow of water pours in at the top bucket of a simple circular symmetrical waterwheel. Each bucket is a leaking one having a hole at its bottom. The behaviour has been extensively described by many authors [2, 5, 6]. In short, we will say that the waterwheel presents four possible states: stationary, rolling always in the same direction, periodic turns in opposite directions, and chaotic turns.

In this paper, we derive the waterwheel equations from the classical mechanics approach. In the continuous mass limit, we derive the Lorenzian equations. The latter approximation becomes feasible with increasing the number of buckets. We compare two cases, small and large number of buckets, and show the difference in the dynamics of their moments of inertia, and, thus, in the overall dynamics.

2 Analytical mechanics of Malkus' waterwheel

Consider a waterwheel with n identical leaky buckets along its rim. Water flows from the top and fills only the bucket i that has an angle θ closest to the top. The wheel with radius R starts rotating and other buckets reach the top and get the water. At time t, each bucket has a mass $m_i(t)$. We also define I_0 as the moment of inertia that the waterwheel would have if none of the buckets had water in them. Throughout, the bucket size l is assumed to be much smaller than the wheel's rim length: $l \ll 2\pi R$. Let us start with the case when $nl < 2\pi R$, with buckets located equidistantly along the rim.

The kinetic and potential energies of the waterwheel, respectively, are

$$T = \frac{1}{2}\left(I_0 + R^2 \sum_{i=0}^{n-1} m_i\right)\dot{\theta}^2 \tag{1}$$

and

$$V = gR\sum_{i=0}^{n-1} m_i \cos\left(\theta + i\frac{2\pi}{n}\right), \tag{2}$$

where g is the gravitational constant.

The dissipative Lagrange equations [1] defines the waterwheel's equation of motion:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = -\frac{\partial \mathcal{F}}{\partial \dot{\theta}}, \tag{3}$$

where the Lagrangian and the Rayleigh's dissipation function, respectively, are

$$L = T - V = \frac{1}{2}(I_0 + R^2\sum_{i=0}^{n-1} m_i)\dot{\theta}^2 - gR\sum_{i=0}^{n-1} m_i \cos\left(\theta + i\frac{2\pi}{n}\right) \tag{4}$$

and

$$\mathcal{F} = \frac{1}{2}\sum_{i=0}^{n-1} \nu\dot{\theta}. \tag{5}$$

Thus, the equation of *motion* is

$$\left(I_0 + R^2\sum_{i=0}^{n-1} m_i\right)\ddot{\theta} + \nu\dot{\theta} - gR\sum_{i=0}^{n-1} m_i \sin\left(\theta + i\frac{2\pi}{n}\right) = 0. \tag{6}$$

The *mass conservation* dynamics is determined by the difference between incoming and outgoing flows:

$$\dot{m_i} = q_i - km_i, \tag{7}$$

where incoming flow rate for i-th cup q_i is

$$q_i = \begin{cases} \tilde{q}, & \text{if } |\theta + i\frac{2\pi}{n}| \leqslant \arcsin\frac{l}{2R}; \\ 0, & \text{otherwise,} \end{cases} \tag{8}$$

Here we assumed that the water jet is narrower than the bucket size l, so that the latter completely defines the angle at which a bucket is exposed to the water.

3 Continuous mass limit: Derivation of Lorenzian system

When the buckets tend to densely cover the whole rim, i.e. $nl \to 2\pi R$, consider the continuous mass approximation:

$$m_i(t) = \int_{\arcsin \frac{l}{2R}}^{-\arcsin \frac{l}{2R}} m(\theta, t)\,\mathrm{d}\theta, \tag{9}$$

with m being the function of t and θ.

In this case, the system's Lagrangian takes the form

$$L = T - V = \frac{1}{2}\Big(I_0 + R^2 \int_0^{2\pi} m\,\mathrm{d}\theta\Big)\dot{\theta}^2 - gR\int_0^{2\pi} m\cos\theta\,\mathrm{d}\theta. \tag{10}$$

It is easy to prove (see below) that the total moment of inertia

$$I = \Big(I_0 + R^2 \int_0^{2\pi} m\,\mathrm{d}\theta\Big) \to \mathrm{const} \tag{11}$$

as $t \to \infty$. Solving system (3) for Lagrangian (10) gives

$$I\ddot{\theta} + \nu\dot{\theta} + gR\int_0^{2\pi}\Big(\frac{\partial m}{\partial\theta}\cos\theta - m\sin\theta\Big)\,\mathrm{d}\theta = 0. \tag{12}$$

Taking $m_h = m\cos\theta$ as the horizontal projection of m, obtain

$$\int_0^{2\pi} \frac{\partial m}{\partial\theta}\cos\theta\,\mathrm{d}\theta = \int_0^{2\pi}\frac{\partial m_h}{\partial\theta}\,\mathrm{d}\theta = m_h\big|_0^{2\pi} = 0, \tag{13}$$

which leaves us with only the terms that accounts for the damping and gravitational torques. Considering $\omega = \dot{\theta}$, obtain the *torque balance* equation:

$$I\dot{\omega} = -\nu\omega + gR\int_0^{2\pi} m\sin\theta\,\mathrm{d}\theta. \tag{14}$$

Since $m(t, \theta)$ is periodic in θ, present it in the form

$$m(\theta, t) = A(t)\sin\theta + B(t)\cos\theta, \tag{15}$$

where $A(t)$ and $B(t)$ are the amplitudes. Then, substitution of (15) into (14) gives the following equation of *motion*:

$$I\dot{\omega} = -\nu\omega + \pi gRA. \tag{16}$$

For an individual mass unit, together with the masses of water falling onto the arc element $\triangle Q$ and leaking out of it, one must consider the change of water in the sector $\triangle Q$ due to the wheel's rotation [5, 6]. The *mass conservation* or the *continuity* equation takes the form:

$$\frac{\partial m}{\partial t} = q - km - \omega \frac{\partial m}{\partial \theta}, \tag{17}$$

where the last term accounts for the water turned in the sector and out of it. Note that $q = q(\theta)$ here.

Considering the total water mass over the wheel, obtain

$$\dot{M} = Q - kM, \tag{18}$$

where the total mass and the total inflow, respectively, are

$$M = \int_0^{2\pi} m(\theta, t)\, \mathrm{d}\theta \quad \text{and} \quad Q = \int_0^{2\pi} q(\theta)\, \mathrm{d}\theta. \tag{19}$$

Stationary point of Eq. (18) defines the moment of inertia caused by the water at $t \to \infty$, so that

$$I = I_0 + R^2 \frac{Q}{k}. \tag{20}$$

Since the water is added symmetrically at the top of the wheel, i.e. the same inflow occurs at θ and $-\theta$, present

$$q(\theta) = \tilde{q} \cos \theta, \tag{21}$$

Substitution of (15) and (21) into (17) gives the following amplitude equations:

$$\dot{A} = -kA + \omega B \tag{22}$$

$$\dot{B} = q - kB - \omega A \tag{23}$$

Introducing new variables x, y, z and new time τ via the relations

$$\omega = kx, \quad A = \frac{\nu k}{\pi g R} y, \quad B = -\frac{\nu k}{\pi g R} z + \frac{\tilde{q}}{k}, \quad t = \frac{\tau}{k}, \tag{24}$$

obtain:

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - z, \end{aligned} \tag{25}$$

where

$$\sigma = \frac{\nu}{Ik} \quad \text{and} \quad r = \frac{\pi g \tilde{q} R}{k^2 \nu}. \tag{26}$$

Equations (25) constitute the *Lorenzian* system with the attractor reminiscent of the famous Lorenz one [6].

Note that all results remain unchanged if one presents $m(\theta, t)$ and $q(\theta)$ by a Fourier series [5]. This implies that the waterwheel dynamics is governed by (25) even if the wheel's plane is tilted. If this is the case, one must consider the effective gravitational constant, $g_{ef} = g \sin \alpha$, where α is the tilt of the wheel from horizontal.

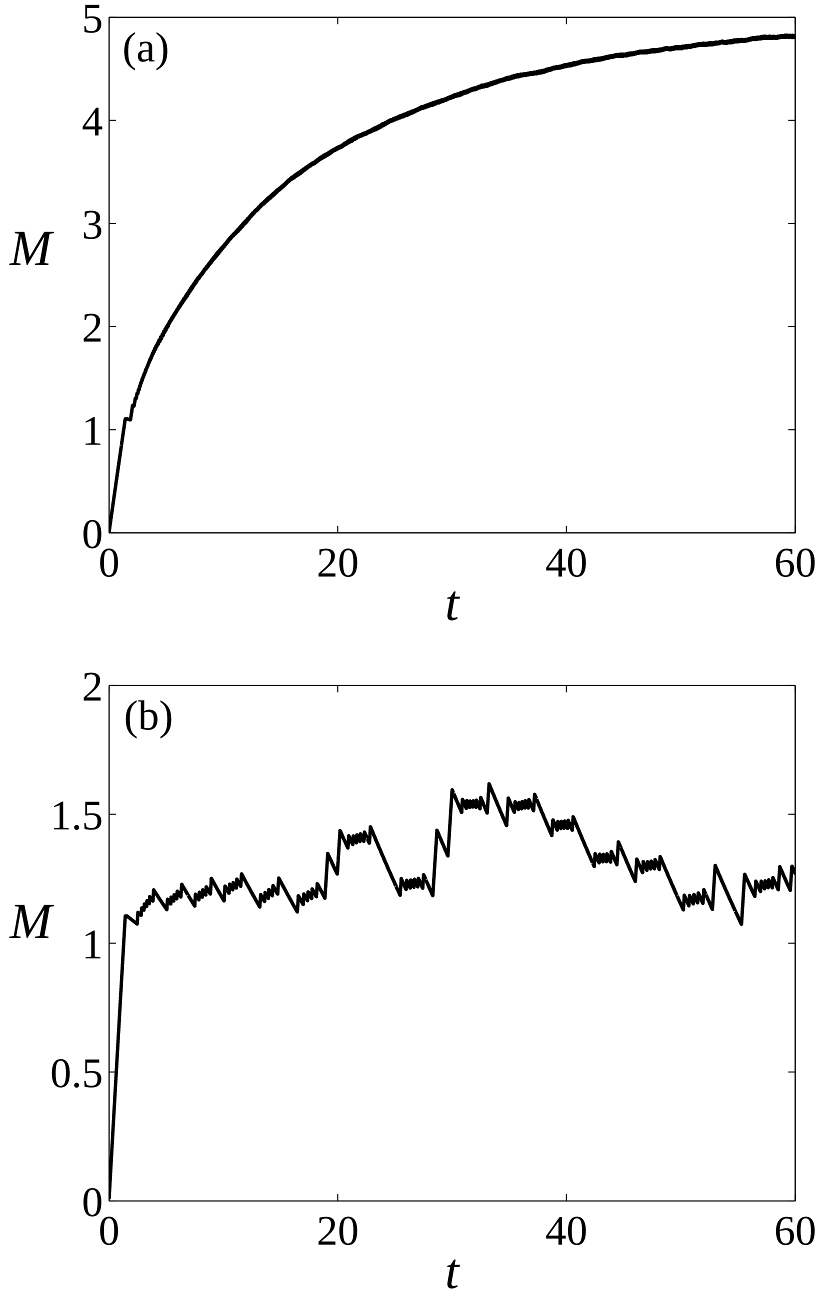

Fig. 1: Dynamics of the total mass of water $M = \sum_{i=0}^{n-1} m_i$ for a) $n = 29$; b) $n = 8$.

4 Dynamics for different number of buckets

Since the Lorenzian system is derived in the continuous mass limit, we would like to know how much Lorenz-like the waterwheel would be for a discrete mass case. In the continuous mass case, the moment of inertia caused by the water, I_w, is proven to tend to $R^2\frac{Q}{k}$ as $t \to \infty$, meaning that, after a certain transient, the wheel's total moment of inertia becomes a constant. Let us consider the dynamics of I_w in the discrete mass case taking the following parameters: $q = 0.8$ kg/s, $k = 0.013\ \mathrm{s}^{-1}$, $I_0 = 0.04\ \mathrm{kg}\cdot\mathrm{m}^2$, $\nu = 1.5\ \mathrm{kg}\cdot\mathrm{m}^2/\mathrm{s}$, $R = 2$ m, $l = 0.435$ m, and $g = 9.8\ \mathrm{m/s}^2$. For the given choice, the maximal number of buckets densely covering the wheel's rim is $n_{max} = \frac{2\pi R}{l} \approx 29$. In this case, the dynamics of total mass perfectly agrees with the theoretical prediction for the continuous mass case: it tends to a constant value (see Fig. 1a). Considering $N = 8$, the total mass of water reaches a quasi-stationary state, resembling the large bucket's dynamics only on the average (see Fig. 1b). Thus, the overall dynamics of two systems will be different.

5 Conclusion

The Lagrange approach is shown to be the framework for deriving the waterwheel's dynamics equations for both discrete and continuous mass cases. The dynamics of discrete mass system becomes more Lorenz-like as the number of buckets increases.

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