# Self-similarity/Memory-length Parameter Estimation for non-Gaussian Hermite Processes via Chaos Expansion

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**Abstract.** We study the behavior of Hermite processes of order $q$ with self-similarity index $H \in (\frac{1}{2}, 1)$. Using Wiener-Itô multiple stochastic integrals and Malliavin calculus we develop a class of consistent estimators for the self-similarity index based on the asymptotic behavior of the filtered process. This research is part of a joint project with Frederi Viens (Purdue) and Ciprian Tudor (Paris Sorbonne).
**Keywords:** Hurst index estimation, Hermite process, Wiener chaos expansion, Malliavin calculus.

## 1 Introduction

A large number of phenomena in nature can be described effectively by self-similar processes. In a self-similar process any part of the trajectory is invariant under time scaling. Consequently observations that are far apart are strongly correlated what is known as long-range dependence. In literature this type of phenomena arise in different scientific fields such as economics, biology, medicine, traffic network, hydrolodgy.

In our work we focus our attention on a special class of self-similar processes that exhibit long-range dependence, the Hermite processes. More specifically we study the behavior of the quadratic variation for such processes and we apply the corresponding results in the estimation of the self-similarity index.

The Hermite process was initially introduced by Taqqu [6] and Dobrushin and Major [4]. It is defined in the following way

**Definition 1.** The Hermite process $(Z_t^{(q,H)})_{t\in[0,1]}$ of order $q \geq 1$ and with self-similarity index $H \in (\frac{1}{2}, 1)$, $t \in [0, 1]$ is given by

$$Z_t^{(q,H)} = d(H) \int_0^t \dots \int_0^t dW_{y_1} \dots dW_{y_q} \left( \int_{y_1 \vee \ldots \vee y_q}^t \partial_1 K^{H'}(u, y_1) \dots \partial_1 K^{H'}(u, y_q) du \right),$$

where the kernel $K^H(t,s)$ is the same as the kernel of the fractional Brownian motion (fBm) $c_H s^{\frac{1}{2}-H}\int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du$ when $t>s$, $c_H = \left(\frac{H(2H-1)}{\beta(2-2H,H-\frac{1}{2})}\right)^{\frac{1}{2}}$, $\beta(\cdot,\cdot)$ is the Beta function and $H' = 1+\frac{H-1}{q}$.

Among the basic properties of this family are the following:

- the Hermite process $Z^{(q,H)}$ is $H$-selfsimilar and has stationary increments.
- the covariance function of $Z^{(q,H)}$ is identical to that of fBm, namely

$$E\left[Z_s^{(q,H)} Z_t^{(q,H)}\right] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}). \tag{1}$$

- the mean square of the increment is given by $\mathbf{E}\left[\left|Z_t^{(q,H)} - Z_s^{(q,H)}\right|^2\right] = |t-s|^{2H}$. As a consequence, it follows from the Kolmogorov's criterion that $Z^{(q,H)}$ has Hölder-continuous paths of any order $\delta < H$.
- it exhibits long-range dependence in the sense that $\sum_{n\geq 1} \mathbf{E}[Z_1^{(q,H)}(Z_{n+1}^{(q,H)} - Z_n^{(q,H)})] = \infty$. In fact, the summand in this series is of order $n^{2H-2}$. This property is identical to that of fBm since the processes share the same covariance structure.
- if $q=1$ then $Z^{(1,H)}$ is a fractional Brownian motion with Hurst parameter $H$ while for $q \geq 2$ this stochastic process is not Gaussian. In the case $q=2$ this stochastic process is known as *Rosenblatt process*.

The Hurst or self-similarity or long-memory parameter $H$ describes the memory and characterizes all the important properties of a Hermite process. Therefore, it is of great interest to estimate it efficiently. So far many estimators have been proposed using other techniques such as maximum likelihood (exact and approximate), variograms, wavelets, spectral methods. A complete reference of those methods can be found in the book of Beran, J. [1]. The approach we consider is based on the discrete variations of the filtered process.

Assume that we observe the process $Z^{(q,H)}$ at discrete times $\{0, \frac{1}{N}, \frac{2}{N}, \ldots, 1\}$ and let $\alpha$ be a filter of length $l+1$ and order $p \geq 1$; that is $\alpha$ is an $l+1$ dimensional vector such that $\sum_{q=0}^{l} \alpha_q q^r$ for $0 \leq r \leq p-1$ and $\sum_{q=0}^{l} \alpha_q q^p \neq 0$. Then the quadratic variation statistic associated with the filter $\alpha$ is defined as

$$V_N(2,\alpha) = \frac{1}{N-l}\sum_{i=l}^{N-1}\left\{\frac{|V_\alpha(i/N)|^2}{E|V_\alpha(i/N)|^2} - 1\right\}, \tag{2}$$

where $|V_\alpha(i/N)| = \sum_{q=0}^{l} \alpha_q Z_{\frac{i-q}{N}}$.

Our analysis is an extension of results for the fractional Brownian motion and Rosenblatt process for a filter of order 1 by Tudor and Viens [7]. We apply

Wiener-Itô chaos expansions as well as recent results on the convergence of multiple integrals in order to study the asymptotic behavior of the proposed estimators.

The article is structured as follows: Section 2 presents a short introduction on fractional analysis. Section 3 presents the main results for the Hermite processes while in section 4 the results for the Rosenblatt data for a higher order filter. Finally, section 5 illustrates the application of our analysis in the estimation of the Hurst parameter.

## 2 Multiplication in Wiener Chaos

Let $(W_t)_{t\in[0,1]}$ be a classical Wiener process on a standard Wiener space $(\Omega, \mathcal{F}, \mathbf{P})$. If $f \in L^2([0,1]^n)$ with $n \geq 1$ integer, we introduce the multiple Wiener-Itô integral of $f$ with respect to $W$. We refer to [5] for a detailed exposition of the construction and the properties of multiple Wiener-Itô integrals.

Let $f \in \mathcal{S}$, i.e. $f$ is an elementary function, meaning that

$$f = \sum_{i_1,\ldots,i_m} c_{i_1,\ldots i_m} 1_{A_{i_i}\times\ldots\times A_{i_m}}$$

where the coefficients satisfy $c_{i_1,\ldots i_m} = 0$ if two indices $i_k$ and $i_l$ are equal and the sets $A_i \in \mathcal{B}([0,1])$ are disjoints.

For a such step function $f$ we define

$$I_n(f) = \sum_{i_1,\ldots,i_m} c_{i_1,\ldots i_m} W(A_{i_1})\ldots W(A_{i_m})$$

where we put $W([a,b]) = W_b - W_a$.

It can be seen that the application $I_n$ constructed above from $\mathcal{S}$ to $L^2(\Omega)$ is an isometry on $\mathcal{S}$ , i.e.

$$\mathbf{E}\left[I_n(f)I_m(g)\right] = n!\langle f, g\rangle_{L^2([0,1]^n)} \text{ if } m = n \tag{3}$$

and

$$\mathbf{E}\left[I_n(f)I_m(g)\right] = 0 \text{ if } m \neq n.$$

It also holds that

$$I_n(f) = I_n\left(\tilde{f}\right)$$

where $\tilde{f}$ denotes the symmetrization of $f$ defined by $\tilde{f}(x_1,\ldots,x_x) = \frac{1}{n!} \sum_{\sigma\in_n} f(x_{\sigma(1)},\ldots,x_{\sigma(n)})$.

Since the set $\mathcal{S}$ is dense in $L^2([0,1]^n)$ the mapping $I_n$ can be extended to an isometry from $L^2([0,1]^n)$ to $L^2(\Omega)$ and the above properties hold true for this extension. Note also that $I_n$ can be viewed as an iterated stochastic integral

$$I_n(f) = n! \int_0^1 \int_0^{t_n} \dots \int_0^{t_2} f(t_{t_1}, \dots, t_n) dW_{t_1} \dots dW_{t_n};$$

here the integrals are of Itô type; this formula is easy to show for elementary $f$'s, and follows for general $f \in L^2([0,1]^n)$ by a density argument.

We recall the product for two multiple integrals: if $f \in L^2([0,1]^n)$ and $g \in L^2([0,1])^m$ then it holds that

$$I_n(f) I_m(g) = \sum_{l=0}^{m \wedge n} l! C_l^m C_l^n I_{m+n-2l}(f \otimes_l g) \tag{4}$$

where the contraction $f \otimes_l g$ belongs to $L^2([0,1]^{m+n-2l})$ for $l = 0, 1, \dots, m \wedge n$ and it is given by

$$\begin{aligned} &(f \otimes_l g)(s_1, \dots, s_{n-l}, t_1, \dots, t_{m-l}) \\ &= \int_{[0,1]^l} f(s_1, \dots, s_{n-l}, u_1, \dots, u_l) g(t_1, \dots, t_{m-l}, u_1, \dots, u_l) du_1 \dots du_l \end{aligned}$$

## 3 Variations of the Hermite Process

In this case we consider a filter of order 1, i.e. $\alpha = \{-1, +1\}$. Therefore, we get that

$$V_\alpha\left(\frac{i}{N}\right) = \left(Z^{(q)}_{\frac{i+1}{N}} - Z^{(q)}_{\frac{i}{N}}\right). \tag{5}$$

The fundamental idea in our approach starts from the fact that we can find an explicit expansion of $V_N$ in the Wiener chaos. Indeed, we know that the increment of the process can be expressed as a Wiener integral

$$Z^{(q,H)}_{\frac{i+1}{N}} - Z^{(q,H)}_{\frac{i}{N}} = I_q(f_{i,N}),$$

where

$$\begin{aligned} &f_{i,N}(y_1, \dots, y_q) = \\ &1_{[0, \frac{i+1}{N}]}(y_1 \vee \dots \vee y_q) d(H) \int_{y_1 \vee \dots \vee y_q}^{\frac{i+1}{N}} \partial_1 K^{H'}(u, y_1) \dots \partial_1 K^{H'}(u, y_q) du \\ &-1_{[0, \frac{i}{N}]}(y_1 \vee \dots \vee y_q) d(H) \int_{y_1 \vee \dots \vee y_q}^{\frac{i}{N}} \partial_1 K^{H'}(u, y_1) \dots \partial_1 K^{H'}(u, y_q) du. \end{aligned}$$

Having this as a starting point the steps we follow are:

1. We can write the Wiener Chaos Expansion of $V_N$:

$$V_N = N^{2H-1} \sum_{k=0}^{q-1} I_{2q-2k} \left( \sum_{i=0}^{N-1} f_{i,N} \otimes_k f_{i,N} \right) = \sum_{n=1}^{q} c_{2n} T_{2n}, \qquad (6)$$

   where $I_{2q-2k}$ is a multiple Wiener-Itô integral with respect to W and $f_{i,N}(y_1, \ldots, y_q)$ defined as above.
2. We estimate the $L^2$ norm of the terms appearing in the chaos expansion of $V_N$. The term $T_2 := N^{2H-1} I_2 \left( \sum_{i=0}^{N-1} f_{i,N} \otimes_{q-1} f_{i,N} \right)$ is the dominant term. Therefore, the behavior of $V_N$ is determined by the behavior of $T_2$. We prove that

$$\lim_{N\to\infty} \mathbf{E} \left[ c_{1,H}^{-1} N^{(2-2H')2} c_2^{-2} V_N^2 \right] = 1.$$

3. We determine the asymptotic distribution of $V_N$. First of all we start by checking if we have convergence to a Normal limit, by using the Nualart Ortiz Latorre criterion, according to which
   - *If* $\{F_N = I_n(f_N) : N \in \boldsymbol{N}\}$ *is a sequence of* $n$*th Wiener chaos random variables and* $E\left[|F_N|^2\right] = ||f_N||^2_{H^{\otimes 2}}$ *converges to 1 as* $N$ *tends to infinity, then* $\lim_{N\to\infty} F_N = \mathcal{N}(0,1)$ *in distribution if and only if* $\lim_{N\to\infty} \mathbf{E}\left[ \left( ||DF_N||_H^2 - n \right)^2 \right] = 0$.

   If this is not satisfied, then we need to identify the limiting distribution. In this case, using the isometry property of the Wiener-Itô integral we prove that the kernel of the term $T_2$ converges to the kernel of the Rosenblatt process at time 1.

We summarize our results in the following theorem:

**Theorem 1.** *Let* $H \in (1/2, 1)$ *and* $q \in \mathbf{N} \setminus \{0\}$. *Let* $Z^{(q,H)}$ *be a Hermite process of order* $q$ *and self-similarity parameter* $H$. *Define*

$$c = (q!)^{-1} \frac{(4H'-3)^{1/2}(4H'-2)^{1/2}}{2d(H)^2 (H'(2H'-1))^{q-1}}.$$

*Then* $cN^{(2-2H)/q} V_N$ *converges in* $L^2(\Omega)$ *to a standard Rosenblatt random variable with parameter* $H'' := \frac{2(H-1)}{q} + 1$*, that is, to the value at time* 1 *of a Hermite process of order 2 and self-similarity parameter* $H''$.

In previous results about the fractional Brownian motion (see [3] and [7]), the limiting distribution was Normal for certain values of $H$. Here instead, we observe that this is not true. The limiting distribution is proved to be Rosenblatt always.

By studying the behavior of the remaining terms in the decomposition of $V_N$ (the non-dominant terms) we discovered what we call the **reproduction property**: *From a Hermite process of order* $q > 2$ *we can construct Hermite processes of lower orders.* This is summarized as follows:

**Theorem 2.** *Let $Z^{(q,H)}$ be again a Hermite process, as in the previous theorem. Let $T_{2n}$ be the term of order $2n$ in the Wiener chaos expansion of $V_N$: this is a multiple Wiener integral of order $2n$, and we write $V_N = \sum_{n=1}^{q} c_{2n}T_{2n}$ where $c_{2q-2k} = k!(C_k^q)^2$.*

- *For every $H \in (1/2, 1)$ and for every $k = 1, \ldots, q-1$ we have convergence of the expression $(z_{k,H})^{-1} N^{(2-2H')k} T_{2k}$ in $L^2(\Omega)$ to $Z^{r,K}$, a Hermite random variable of order $r = 2k$ with self-similarity parameter $K = 2k(H'-1)+1$.*
- *For every $H \in (1/2, 3/4)$, with $k = q$, we have convergence of $x_{1,H}^{-1/2}\sqrt{N}T_{2q}$ to a standard normal distribution.*
- *For every $H \in (3/4, 1)$, with $k = q$, we have convergence of $x_{2,H}^{-1/2}N^{2-2H}T_{2q}$ in $L^2(\Omega)$ to $Z^{2q,2H-1}$, a Hermite r.v. of order $2q$ with parameter $2H-1$.*
- *For $H = 3/4$, with $k = q$, we have convergence of $\sqrt{N/\log N}x_{3,H}^{-1/2}T_{2q}$ to a standard normal distribution.*

*The terms $z_{k,H}$, $x_{1,H}$, $x_{2,H}$, $x_{3,H}$ are constants that depend only on $H$.*

More details can be found in [2].

## 4 Variations of the Rosenblatt Process

The Rosenblatt process is a Hermite process of order $q = 2$. Therefore, it admits the following representation

$$Z(t) = \int_0^t \int_0^t L_t(y_1, y_2) dW_{y_1} dW_{y_2}, \tag{7}$$

where $L_t(y_1, y_2)$ is the kernel of the Rosenblatt process

$$L_t(y_1, y_2) = d(H) \int_{y_1 \vee y_2}^t \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) du \tag{8}$$

and $H' = \frac{H+1}{2}$ and $d(H) = \frac{1}{H+1}\left(\frac{H}{2(2H-1)}\right)^{-1/2}$. In this situation we are interested in studying the behavior of the statistic for a longer filter. For simplicity, we illustrate the results using a filter of order 2 (i.e. $\alpha = \{-1, 0, +1\}$). However, an extension for a filter of higher order is straightforward. Observe that

$$E\left[Z\left(\frac{i+1}{N}\right) - 2Z\left(\frac{i}{N}\right) + Z\left(\frac{i-1}{N}\right)\right]^2 = (4-4^H)N^{-2H}.$$

Therefore, the variation statistic (2) becomes

$$\frac{N^{2H-1}}{(4-4^H)} \sum_{i=1}^{N} \left\{ \left[Z\left(\frac{i+1}{N}\right) - 2Z\left(\frac{i}{N}\right) + Z\left(\frac{i-1}{N}\right)\right]^2 - (4-4^H)N^{-2H} \right\}$$

Using the Wiener chaos expansion of the statistic and the same procedure as the one described above, we estiblish the following

**Theorem 3.** *The quantity* $c_H N^{1-H} V_N(2, \{-1, 0, +1\})$ *converges in* $L^2(\Omega)$ *to a standard Normal random variable as* $N \to \infty$.

Therefore, we observe that as in the fractional Brownian motion case for $H > 3/4$ (see [7] for details) we obtain convergence to a normal limit when we use a longer filter on our data, or in other words when we keep differentiating the process.

## 5 Estimators for the self-similarity index $H$

In this section we construct an estimator for the self-similarity index based on the discrete observations of the process. We illustrate how this procedure works for a Hermite process and using a filter of order 1.

- Denote by $S_N(2, \alpha)$ the average of the squared increment i.e.

$$S_N(2, \alpha) = \frac{1}{N} \sum_{i=0}^{N-1} \left( Z^{(q,H)}_{\frac{i+1}{N}} - Z^{(q,H)}_{\frac{i}{N}} \right)^2$$

- Recall that $E[S_N(2, \alpha)] = N^{-2H}$
- Estimate $S_N(2, \alpha)$ by its mean,

$$S_N(2, \alpha) = N^{-2\hat{H}_N(2,\alpha)}$$

- Solve with respect to $\hat{H}_N(2, \alpha)$ in order to get the estimate for the filter with respect to $S_N(2, \alpha)$. In the case of filter of order 1 it is straightforward and it is equal to $\hat{H}_N = (\log S_N) / (2 \log N)$. For higher order filter we need to solve a nonlinear equation (which can be easily done numerically).
- Therefore, using the results about the variations of the Hermite process we can obtain the limiting distribution of our estimator.
- Finally, it can be shown that the estimator is strongly consistent.

We state the theorem for the case of a Hermite process of order $q$ for filter of order 1.

**Theorem 4.** *The estimator* $\hat{H}_N$ *is strongly consistent, i.e.* $\lim_{N\to\infty} \hat{H}_N = H$ *almost surely. Moreover there exists a standard Rosenblatt random variable* $R$ *with self-similarity parameter* $1 + 2(H-1)/q$ *such that*

$$\lim_{N\to\infty} \mathbf{E}\left[ \left| 2N^{2\left(1-\hat{H}_N\right)/q} \left( H - \hat{H}_N \right) \log N - q! c_{1,H,q}^{1/2} R \right| \right] = 0,$$

*where* $c_{1,H,q}$ *s a constant that depends on* $q$ *and* $H$.

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