# The non-viscous Burgers equation associated with random positions in coordinate space: a threshold for blow up behaviour[⋆]

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It is well known that the solutions to the non-viscous Burgers equation

$$u_t + (u, \nabla)u = -\beta u, \tag{1}$$

where $u(x,t) = (u_1, ..., u_n)$ is a vector-function $\mathcal{R}^{n+1} \to \mathcal{R}^n$, $\beta \geq 0$ is a constant friction coefficient develop a gradient catastrophe at a critical time $T$ provided the initial data have a negative derivative later then $-\beta$ in certain points. Our main question is: can a stochastic perturbation suppress the appearance of unbounded gradients?

We can introduce the Lagrangian coordinate $x(t)$ to label a point which moves together with the medium, that is $\frac{dx(t)}{dt} = u(t, x(t)) := u_1(t)$. Thus, $x = x(t)$ is the equation for the particle path, when the particle moves along the Burgers fluid. Equation (1) is equivalent to the following system of ODE:

$$\dot{x}(t) = u_1(t), \quad \dot{u}_1(t) = -\beta u_1(t) \tag{2}$$

Further on we will omit the index 1. In the theory of stochastic dynamical systems often consider a stochastic perturbation of the velocity, which leads to the appearance of a white noise in the second of equations (2). We consider this equation assuming that the particle paths in the medium are governed by a random process with a variance which depends in a polynomial way on the velocity. Namely, we consider a medium with random particles paths, more precisely, described by a $2 \times n$ dimensional Itô stochastic differential system of equations

$$dX_k(t) = U_k(t), dt + \sigma |U(t)|^p, d(W_k)_t, \quad X_0 = x,$$

$$dU_k(t) = -\beta U_k(t) dt, \quad k = 1, .., n, \tag{3}$$

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$$X(0) = x, \quad U(0) = u, \quad t \ge 0,$$

where $(X(t), U(t))$ runs in the phase space $\Omega \times \mathcal{R}^n$, $,\Omega \subset \mathcal{R}^n$,$,\sigma > 0$ and $p \ge 0$ are constants, $(W)_t = (W)_{k,t}, k = 1, ..., n$, is the $n$ - dimensional Brownian motion.

We can interpret system (3) also as follows: assume that we measure the position of a particle with an error depending on its velocity and then try to restore the velocity. Can we hope to extract from our measurement a realistic information on such critical phenomena as the blow up occurring in a medium described by the Burgers equation associated with (3)? As we will see, the answer depends on the exponent $p$ – e.g., if $p > 1$, the information gets lost.

Given an initial distribution of the particles which is uniform in space and with the initial velocity linearly depending on the position we show both analytically and numerically that there exists a threshold effect: if the power in the above variance is less or equal 1, then the noise does not influence the solution behavior, in the following sense: the mean of the velocity when we keep the value of position fixed (denoted as $\hat{u}(t,x)$) goes to infinity outside the origin. If however the power is larger than 1, then this mean decays to zero as the time tends to a critical value. For linear initial data $u_0(x) = \alpha x, \alpha < 0$, we get the following result.

**Theorem 1.** *If initially the particles are distributed uniformly in the bounded domain $\Omega_L \subset \mathcal{R}^n$, the asymptotic behaviour of $\hat{u}(t,x)$ for $t \to T$ can be analyzed explicitly.*

*Namely, for any $p \in [0,1]$ the mean $\hat{u}(t,x)$, being equal to zero at any point $x \in \mathcal{R}^n, \quad t = T$, is discontinuous at every such point if $x \ne 0$. More precisely, if $p = 0$, $\hat{u}(t,x)$ coincides with the solution to (1) for $t < T$. For $p \in (0,1]$ the asymptotics*

$$\hat{u}(t,x) = \frac{\alpha}{1-\frac{t}{T}} x + o\left(\frac{1}{1-\frac{t}{T}}\right), \quad t \to T, \quad x \in \mathcal{R}^n,$$

*takes place.*

*For $p > 1$ at any $x \in \mathcal{R}^n, x \ne 0$, $|\hat{u}(t,x)| \to 0$ as $t \to T$, however, $\mathrm{div}_x \hat{u}(t,x) \to \infty, x = 0$, $t \to T$.*
*More precisely, for any $x \ne 0$,*

$$\hat{u}(t,x) = -C|x|^{\frac{2(1-p)}{p}} x \left(1 - \frac{t}{T}\right) + o(1 - \frac{t}{T}), \quad t \to T.$$

*where $C$ is the positive constant, and for any $t \in [0,T)$*

$$\hat{u}(t,x) = \frac{\alpha}{1-\frac{t}{T}} x + o\left(\frac{1}{1-\frac{t}{T}}\right), \quad x \to 0.$$